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OPTIMAL CONTROL ANALYSIS OF SHIGELLA EPIDEMIC MODEL

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ABSTRACT

In this paper, a model for the transmission dynamics of shigella is formulated and five control strategies: vaccination, education campaigns, screening, treatment and sanitation are deployed to minimize the total number of infected individuals, number of bacteria population and the cost associated with the control strategies. Optimal control theory is applied to a system of ordinary differential equations of a shigella epidemic. The Pontryagin's maximum principle is employed to find the necessary and sufficient conditions for the existence of the optimal controls. Runge-Kutta forward-backward sweep numerical approximation method is used to solve the optimal control system. The results show that for the shigella outbreak to be under control in the community, 49.995% *of the vaccine,* 49.995% *of education campaign,* $8.4206\times10^{-3}\%$ *of screening,* $3.2323\times10^{-12}\%$ *of treatment and* $1.5794\times10^{-3}\%$ *of sanitation should be continually implemented.*

Keywords: SVGEIAHRB Model, Shigella, Pontryagin's Maximum Principle, Optimal Control, Vaccination, Treatment, Education Campaign, Screening, Hamiltonian, Numerical Simulation, Transmission.

INTRODUCTION

Shigella infection is a diarrheal infection caused by *Shigella* species that is gram-negative, non-spore forming, facultative anaerobes that infect the intestinal lining and spread by fecal-oral transmission.

Optimal control theory is a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function (performance index) is optimized (Ross, 2015). The optimal control (OC) can also be defined as the process of determining the control and state trajectories for a dynamic system over a period of time in order to minimize a performance index. It deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost function. It can be derived using Pontryagin's maximum principle (a necessary condition also known as Pontryagin's minimum principle or simply Pontryagin's Principle) (Ross, 2009; Berhe *et al.*, 2018; Berhe *et al.*, 2019 and Edward *et al.*, 2020b) or by solving the Hamiltonian equation for sufficient condition.

The application of optimal control theory has become another interesting area in the field of mathematical modeling (Lenhart, 2007) in that it provides insightful understanding of many

biomedical problems and its being used extensively in the controlling of infectious diseases (Kirschner et al., 1997; Lenhart, 2007). It is mostly used in the control of the spread of numerous diseases for which control measures are in place, for example vaccination, treatment, isolation among other (Nanda et al., 2007; Tunde et al., 2012).
Tunde et al., 2012, applied optimal control to find the optimal combination of vaccination and treatment strategies that will minimize the cost of the two control measures as well as the number of infectives. Kbenesh et al., 2009, also applied optimal control when looking at time dependent prevention and treatment efforts. Again the work of Neilan and Lenhart (2010) serves as an introduction to the theory of optimal control applied to systems of ordinary differential equations with emphasis on disease models. They outline the steps in formulating an optimal control problem and derive necessary conditions. Several simple examples are provided with a detailed methodology in characterizing the optimal control through use of Pontryagin Maximum Principle.

Devipriya and Kalaivani (2012) presented their work on"optimal control of multiple transmission of water-borne disease". A controlled SIWR model was considered which an extension of the simple SIR model was by adjoining a compartment W that tracks the pathogen concentration in the water. The controls represented an immune boosting and pathogen suppressing drugs. Their objective function was based on a combination of minimizing the number of infected individuals and the cost of the drugs dose.

The Hamiltonian is a function used to solve a problem of optimal control for a dynamical system. It can be seen as an instantaneous increment of the Lagrangian expression of the problem that is to be optimized over a certain time horizon (Ferguson and Lim, 1998). The Hamiltonian of optimal control theory was developed by Lev Pontryagin as part of his maximum principle (Dixit, 1990). Pontryagin proved that a necessary condition for solving the optimal control problem is that the control should be chosen so as to optimize the Hamiltonian (Kirk, 1970).

Mathematical formulation of optimal control problem

We formulate an optimal control model for shigella infection (shigellosis) disease in order to derive five optimal control measures with minimal implementation cost to eradicate the disease after a limited period of time. We employ the control efforts $u_i(t); i = 1,2,3,4,5$ in human and animal reservoirs populations and $(1 - u_i(t))$ are the failure rate for the control efforts for $i = 1,2,3,4,5$ .We let $u_1(t)$ be the effort of using vaccines as a control measure, $u_2(t)$ be the efforts of using education campaign as a control measure, $u_3(t)$ be the efforts of using screening as a control measure, $u_4(t)$ be the efforts of using treatment as a control measure and $u_5(t)$ be the efforts of using germicide as a control measure through spraying to decontaminate the environment.

Assumptions

The following model assumptions are made.
(a) We introduced
  (i) vaccination to the susceptible individuals at a rate u_1 such that $u_1 S(t)$ individuals per time are removed from the susceptible class.
  (ii) education campaign to the susceptible individuals at a rate u_2 such that $u_2 S(t)$

individuals per time are removed from the susceptible class.

(iii) screening to the exposed individuals at a rate u_3 such that $u_3E(t)$ individuals per time are removed from the exposed class.

(iv) treatment to the hospitalized individuals at a rate u_4 such that $u_4H(t)$ individuals per time are removed from the hospitalized class.

(v) spraying of germicide to the environment at a rate u_5 such that $u_5B(t)$ bacteria per time are removed from the environmental class.

(b) The infected individuals who cannot recover naturally move to the hospital for treatment.

Flow diagram of the model with control variables

We demonstrate the dynamical transfer of the population with the flow diagram in Figure 1 below.

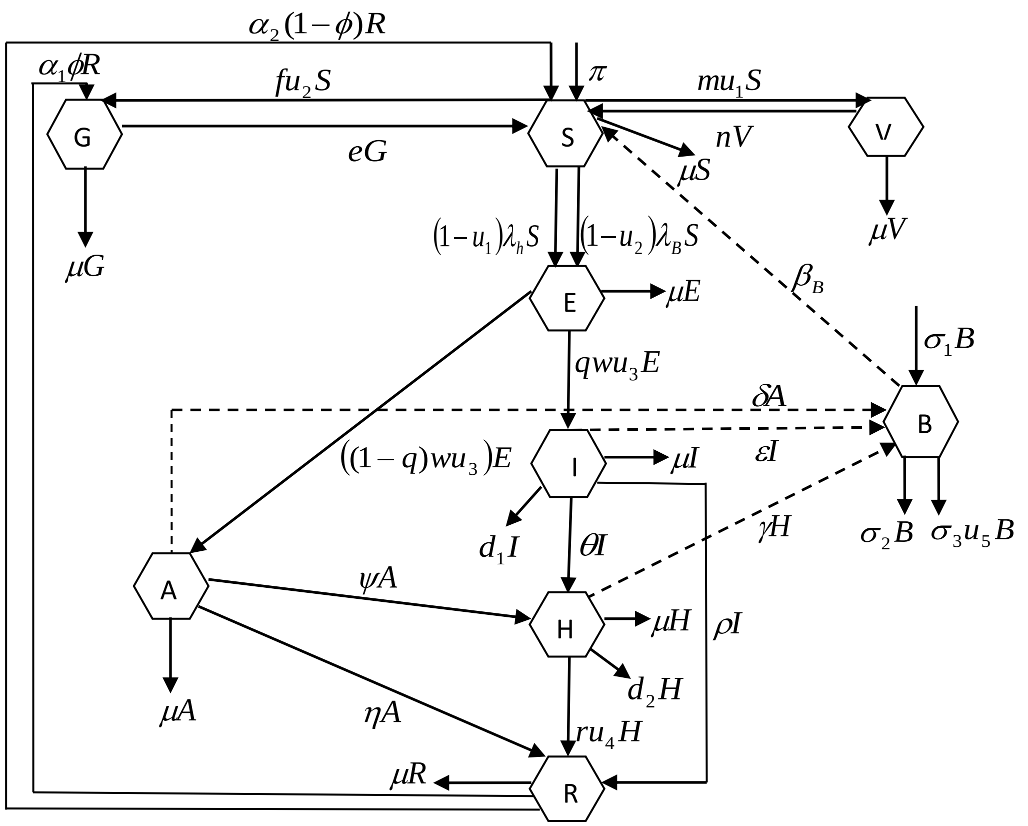

Figure 1. A schematic representation of flow of individuals (solid lines) among states and control variables and flow of Pathogens in the environment (dotted lines) for the environmental infection transmission system (EITS) of the model.

		Optimal control variables of the model
u_1	=	The effort of using vaccines as a control measure by vaccinating the susceptible individuals.
u_2	=	The effort of using education campaign as a control measure by educating the susceptible individuals.
u_3	=	The effort of using screening as a control measure by screening the exposed

individuals.

u_4 = The effort of using treatment as a control measure by treating the asymptomatic, infected and hospitalized individuals.

u_5 = The effort of using germicide as a control measure by decontaminating the contaminated environment.

Table 1: Description of the variables of the models

Variables	Description	Values	Reference
$S(0)$	Number of susceptible individuals at time (t).	2000	Assumed
$V(0)$	Number of vaccinated individuals at time (t).	120	Assumed
$G(0)$	Number of educated individuals at time (t).	150	Assumed
$E(0)$	Number of exposed individuals at time (t).	40	Assumed
$A(0)$	Number of asymptomatic individuals at time (t).	90	Assumed
$I(0)$	Number of infected individuals at time (t).	30	Assumed
$H(0)$	Number of hospitalized individuals at time (t).	60	Assumed
$R(0)$	Number of Recovered individuals at time (t).	200	Assumed
$B(0)$	Number of bacteria in the environment at time (t).	300	Assumed

Table 2: Description of the parameters of the models

Parameters	Description	Values	Reference
π	The recruitment rate.	500	(Edward *et al.*, 2018)
m	The vaccination rate at which the susceptible individuals move to the vaccinated class.	0.02	Assumed
n	The vaccine immunity loss rate at which the vaccinated individuals move to the susceptible class.	0.027	Assumed
f	The education rate at which the susceptible individuals move to the educated class.	0.7	Assumed
e	The recovering rate at which the educated individuals (who failed to adhere to the education they received) moved back to the susceptible class.	0.42	Assumed
r	The rate at which the hospitalized individuals moved to the recovered class.	0.06	Assumed
θ	The rate at which the infected individuals moved to the hospitalized class.	0.03	Assumed
η	The rate at which the asymptomatic individuals moved to the recovered class.	0.41	Assumed

μ	The natural death rate.	0.45	Assumed
d_1	The death rate due to the disease in the infected class.	0.02	(Edward *et al.*, 2018)
d_2	The death rate due to the disease in the hospitalized class.	0.025	Assumed
ϕ	The proportion of the recovered individuals who moved to the educated class at a rate α_1.	0.029	Assumed
$(1-\phi)$	The proportion of the recovered individuals who moved to the susceptible class at a rate α_2.	0.971	Assumed
q	The proportion of the exposed individuals who moved to the infected class at a rate ω.	0.9	(Edward *et al.*, 2018)
$(1-q)$	The proportion of the exposed individuals who moved to the asymptomatic class at a rate.	0.1	Assumed
ω	The incubation rate (rate at which exposed individuals, $E(t)$, progress to either asymptomatic class $A(t)$ or infected $I(t)$.	0.35	(Edward *et al.*, 2018)
ψ	The rate at which the asymptomatic individuals moved to the hospitalized class.	0.04	Assumed
ρ	The recovering rate at which the infected individuals moved to therecovered class.	0.14	Assumed
K	The concentration of *Shigella* in the environment that yields 50% chance of catching dysentery diarrhea (Berhe *et al.*, 2008).	600	Assumed
β_1	The transmission rate of shigella for the infected individuals due to human to human interaction.	0.095	Assumed
β_2	The transmission rate of shigella for the asymptomatic individuals due to human to human interaction.	0.075	Assumed
β_3	The transmission rate of shigella for the hospitalized individuals due to human to human interaction.	0.055	Assumed
β_B	The ingestion rate of shigella by human from the environment.	0.00039	Assumed
ε	Shigella pathogen shedding rate for the infected individuals.	80	(Edward *et al.*, 2018)
δ	Shigella pathogen shedding rate for the asymptomatic individuals.	70	(Edward *et al.*, 2018)

γ	Shigella pathogen shedding rate for the hospitalized individuals.	90	Assumed
σ_1	Shigella pathogen growth rate.	0.73	Assumed
σ_2	Shigella pathogen natural death rate.	0.83	(Edward *et al.*, 2018)
σ_3	Death rate of shigella pathogen due to environmental decontamination.	1.60	Assumed

Equations of the modified model with control variables

The shigella disease is modeled with a system of ordinary differential equations with the five control variables embedded within the dynamical system as stated below:

$$\frac{dS}{dt} = \pi + nV + eG + \alpha_2(1-\phi)R - \left((1-u_1)\lambda_h + (1-u_2)\lambda_B\right)S - \mu S - mu_1 S - fu_2 S \quad (1)$$

$$\frac{dV}{dt} = -(n+\mu)V + mu_1 S \quad (2)$$

$$\frac{dG}{dt} = \alpha_1 \phi R - (e+\mu)G + fu_2 S \quad (3)$$

$$\frac{dE}{dt} = \left((1-u_1)\lambda_h + (1-u_2)\lambda_B\right)S - (wu_3 + \mu)E \quad (4)$$

$$\frac{dA}{dt} = (1-q)wu_3 E - (\eta + \psi + \mu)A \quad (5)$$

$$\frac{dI}{dt} = qwu_3 E - (\theta + \rho + d_1 + \mu)I \quad (6)$$

$$\frac{dH}{dt} = \theta I + \psi A - (ru_4 + d_2 + \mu)H \quad (7)$$

$$\frac{dR}{dt} = ru_4 H + \eta A + \rho I - \alpha_1 \phi R - \alpha_2(1-\phi)R - \mu R \quad (8)$$

$$\frac{dB}{dt} = \varepsilon I + \delta A + \gamma H + (\sigma_3 u_5 + \sigma_2 - \sigma_1)B \quad (9)$$

$N = S + V + G + E + I + A + H + R$ and B

with initial conditions

$S(0) = S_0 > 0, V(0) = V_0 \geq 0, E(0) = E_0 \geq 0, G(0) = G_0 \geq 0, I(0) = I_0 \geq 0, A(0) = A_0 \geq 0,$

$H(0) = H_0 \geq 0, R(0) = R_0 \geq 0,\ B(0) = B_0 > 0, u_1(0) = u_{10},\ u_2(0) = u_{20},\ u_3(0) = u_{30},\ u_4(0) = u_{40},$

$u_5(0) = u_{50}$.

The force of infection for human to human interaction (λ_h) and the force of infection for environment to human interaction (λ_B) are respectively given by:

$$\lambda_0 = \lambda_h + \lambda_B = \beta_1 I + \beta_2 A + \beta_3 H + \frac{\beta_B B}{K+B} \tag{10}$$

The objective functional is defined as follows

$$J(u_1,u_2,u_3,u_4,u_5) = \int_0^T \left(P_1 A + P_2 I + P_3 H + P_4 B + \frac{1}{2}\left(Q_1 u_1^2 + Q_2 u_2^2 + Q_3 u_3^2 + Q_4 u_4^2 + Q_5 u_5^2\right)\right) dt \tag{11}$$

subject to

$$\left.\begin{aligned}
\frac{dS}{dt} &= \pi + nV + eG + \alpha_2(1-\phi)R - \left((1-u_1)\lambda_h + (1-u_2)\lambda_B\right)S - \mu S - mu_1 S - fu_2 S \\
\frac{dV}{dt} &= -(n+\mu)V + mu_1 S \\
\frac{dG}{dt} &= \alpha_1 \phi R - (e+\mu)G + fu_2 S \\
\frac{dE}{dt} &= \left((1-u_1)\lambda_h + (1-u_2)\lambda_B\right)S - (wu_3 + \mu)E \\
\frac{dA}{dt} &= (1-q)wu_3 E - (\eta + \psi + \mu)A \\
\frac{dI}{dt} &= qwu_3 E - (\theta + \rho + d_1 + \mu)I \\
\frac{dH}{dt} &= \theta I + \psi A - (ru_4 + d_2 + \mu)H \\
\frac{dR}{dt} &= ru_4 H + \eta A + \rho I - \alpha_1 \phi R - \alpha_2(1-\phi)R - \mu R \\
\frac{dB}{dt} &= \varepsilon I + \delta A + \gamma H - (\sigma_3 u_5 + \sigma_2 - \sigma_1)B
\end{aligned}\right\} \tag{12}$$

with initial conditions

$S(0) = S_0 > 0$, $V(0) = V_0 \geq 0$, $E(0) = E_0 \geq 0$, $G(0) = G_0 \geq 0$, $I(0) = I_0 \geq 0$, $A(0) = A_0 \geq 0$,

$H(0) = H_0 \geq 0$, $R(0) = R_0 \geq 0$, $B(0) = B_0 > 0$, $u_1(0) = u_{10}$, $u_2(0) = u_{20}$, $u_3(0) = u_{30}$, $u_4(0) = u_{40}$,

$u_5(0) = u_{50}$.

and with control set defined as;

$$U = \begin{cases} (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)) : 0 \le u_1(t) \le u_{1\max} \le 1,\ 0 \le u_2(t) \le u_{2\max} \le 1, \\ 0 \le u_3(t) \le u_{3\max} \le 1,\ 0 \le u_4(t) \le u_{4\max} \le 1,\ 0 \le u_5(t) \le u_{5\max} \le 1 \\ \text{are Lebesgue measureable and piecewise continous on } [0, \mathrm{T}] \end{cases} \quad (13)$$

where $\left((1-u_1)(\beta_1 I + \beta_2 A + \beta_3 H)\lambda_h + (1-u_2)\left(\frac{\beta_B B}{K+B}\right)\lambda_B\right)$ is the reduction in transmission due to vaccine and education campaign; P_1, P_2, P_3 and P_4 are the weight constants or balance factor (i.e. to keep balance) in the size of asymptomatic, infected, hospital and the bacteria population respectively (Zaman *et al.*, 2008) , whereas Q_1, Q_2, Q_3, Q_4 and Q_5 are the weights of control variables $u_i(t)$ relative to its cost implications for vaccination, education campaigns, screening, treatment and sanitation efforts respectively which regulate the optimal control, the square term of the controls indicates the nonlinearity in the cost of controls, and the half-term minimizes the effect of applying the controls (Asamoah *et al.*, 2017). Also, the positive constants Q_1, Q_2, Q_3, Q_4 and Q_5 which are connected with $u_i(t)$ are weight values such that $0 < Q_1, Q_2, Q_3, Q_4 < N$ and $0 < Q_5 < B$. The relative weights given to the positive constants associated with the control terms indicate greater or lower importance placed on minimizing the cost of a control measure (Anita *et al.*, 2011). We also assume that the cost of vaccination, health education campaigns, screening, treatment and spray of germicide is non linear and quadratic as seen in the cost function in (12). $Q_1u_1^2$ represents the cost of vaccination, $Q_2u_2^2$ represents health education campaigns, $Q_3u_3^2$ represents the cost of screening, $Q_4u_4^2$ represents the cost of treatment and $Q_5u_5^2$ represents the cost of spraying germicide in the environment.

Our main goal is to characterize an optimal control $(u_1^*, u_2^*, u_3^* u_4^*, u_5^*) \in U$ which minimizes the cost of vaccination, education campaigns, screening, treatment and sanitation as well as minimizing the number of infectives at terminal time (T) such that the number of susceptible individuals increases. Therefore, to obtain the optimal solutions, we defined the Lagrangian L for the control problem in (13) as follows:

$$L(N, B, u) = P_1A + P_2I + P_3H + P_4B + \frac{1}{2}\left(Q_1u_1^2 + Q_2u_2^2 + Q_3u_3^2 + Q_4u_4^2 + Q_5u_5^2\right), \quad (14)$$

where N is the total population of individuals and B is the pathogen environment. Thus we define our Hamiltonian function H_1 for our control problem as:

$$\begin{aligned} H_1(N, B, u; t) &= P_1A + P_2I + P_3H + P_4B + \frac{1}{2}\left(Q_1u_1^2 + Q_2u_2^2 + Q_3u_3^2 + Q_4u_4^2 + Q_5u_5^2\right) \\ &+ \lambda_1\left(\pi + nV + eG + \alpha_2(1-\phi)R - ((1-u_1)\lambda_h + (1-u_2)\lambda_B)S - \mu S - mu_1S - fu_2S\right) \\ &+ \lambda_2\left(-(n+\mu)V + mu_1S\right) \\ &+ \lambda_3\left(\alpha_1\phi R - (e+\mu)G + fu_2S\right) \\ &+ \lambda_4\left(((1-u_1)\lambda_h + (1-u_2)\lambda_B)S - (wu_3 + \mu)E\right) \end{aligned}$$

$$
\begin{aligned}
&+\lambda_5\left((1-q)wu_3E-(\eta+\psi+\mu)A\right)\\
&+\lambda_6\left(qwu_3E-(\theta+\rho+d_1+\mu)I\right)\\
&+\lambda_7\left(\theta I+\psi A-(ru_4+d_2+\mu)H\right)\\
&+\lambda_8\left(ru_4H+\eta A+\rho I-\alpha_1\phi R-\alpha_2(1-\phi)R-\mu R\right)\\
&+\lambda_9\left(\varepsilon I+\delta A+\gamma H-(\sigma_3u_5+\sigma_2-\sigma_1)B\right) \qquad (15)
\end{aligned}
$$

Model analysis
Optimal Control Solution
We apply the Pontryagin's Maximum Principle (PMP) (Pontryagin *et al.*, 1962) because it is a constrained control problem. The principle identifies the adjoint functions of the optimal system and represents an optimal control in terms of the state and adjoint functions. The main goal of this principle (PMP) is to minimize the objective function. Thus depending on the constraints in the objective function, we want to minimize the Hamiltonian with respect to the controls u_1, u_2, u_3, u_4 and u_5. We define the adjoint functions as λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, λ_7, λ_8 and λ_9 associated with state equations defined for S, V, G, E, A, I, H, R and B.

Theorem: Given that $U^*(t)=\left(u_1^*,u_2^*,u_3^*,u_4^*,u_5^*\right)$ is an optimal control and that $X^*(t)=\left(S^*(t),V^*(t),G^*(t),E^*(t),A^*(t),I^*(t),H^*(t),R^*(t),B^*(t)\right)$ is an optimal state solution for the dynamical optimal control problem (12) – (13) that minimize $J(u_1,u_2,u_3,u_4,u_5)$ over U, there exist adjoint variables $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, $\lambda_4(t)$, $\lambda_5(t)$, $\lambda_6(t)$ $\lambda_7(t)$ $\lambda_8(t)$ and $\lambda_9(t)$ that satisfies the dynamical model below.

$$\frac{d\lambda_1}{dt}=(\lambda_1-\lambda_4)\left((1-u_1)(\beta_1I+\beta_2A+\beta_3H)+(1-u_2)\left(\frac{\beta_BB}{K+B}\right)\right)S+(\lambda_1-\lambda_2)mu_1+(\lambda_1-\lambda_3)fu_2+\lambda_1\mu \qquad (16)$$

$$\frac{d\lambda_2}{dt}=-\lambda_1n+(n+\mu)\lambda_2 \qquad (17)$$

$$\frac{d\lambda_3}{dt}=-e\lambda_1+(e+\mu)\lambda_3 \qquad (18)$$

$$\frac{d\lambda_4}{dt}=(\lambda_4-\lambda_5)wu_3+(\lambda_5-\lambda_6)qwu_3+\lambda_4\mu \qquad (19)$$

$$\frac{d\lambda_5}{dt}=-P_1+(\lambda_5-\lambda_8)\eta+(\lambda_5-\lambda_7)\psi+(\lambda_1-\lambda_4)(1-u_1)\beta_2S-\delta\lambda_9+\mu\lambda_5 \qquad (20)$$

$$\frac{d\lambda_6}{dt}=-P_2+(\lambda_6-\lambda_7)\theta+(\lambda_6-\lambda_8)\rho+(\lambda_1-\lambda_4)(1-u_1)\beta_1S+(d_1+\mu)\lambda_6-\varepsilon\lambda_9 \qquad (21)$$

$$\frac{d\lambda_7}{dt}=-P_3+(\lambda_7-\lambda_8)ru_4+(\lambda_1-\lambda_4)(1-u_1)\beta_3S+(d_2+\mu)\lambda_7-\gamma\lambda_9 \qquad (22)$$

$$\frac{d\lambda_8}{dt}=(\lambda_8-\lambda_1)\alpha_2(1-\phi)+(\lambda_8-\lambda_3)\alpha_1\phi+\mu\lambda_8 \qquad (23)$$

$$\frac{d\lambda_9}{dt}=-P_4+(\lambda_1-\lambda_4)(1-u_2)\left(\frac{K\beta_BS}{(K+B)^2}\right)+(\sigma_3u_5+\sigma_2-\sigma_1)\lambda_9 \qquad (24)$$

with transversality conditions
$\lambda_1(T)=0,\ \lambda_2(T)=0,\ \lambda_3(T)=0,\ \lambda_4(T)=0,\ \lambda_5(T)=0,\ \lambda_6(T)=0,\ \lambda_7(T)=0,\ \lambda_8(T)=0$ and $\lambda_9(T)=0$.
with control functions $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*)$ which satisfies the optimality condition given by:

$$u_1^* = \frac{((\lambda_4-\lambda_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(\lambda_1-\lambda_2)m)S}{Q_1} \tag{25}$$

$$u_2^* = \frac{\left((\lambda_4-\lambda_1)\left(\frac{\beta_B B}{K+B}\right)+(\lambda_1-\lambda_3)f\right)S}{Q_2} \tag{26}$$

$$u_3^* = \frac{(\lambda_4-\lambda_5)wE+(\lambda_5-\lambda_6)qwE}{Q_3} \tag{27}$$

$$u_4^* = \frac{(\lambda_7-\lambda_8)rH}{Q_4} \tag{28}$$

$$u_5^* = \frac{\lambda_9\sigma_3 B}{Q_5} \tag{29}$$

Proof:
In order to obtain the co-state dynamical model and the transversality conditions, we need to apply the Pontryagin's maximum principle (PMP) (Pontryagin *et al.*, 1962) to partially differentiate the Hamiltonian function H_1 with respect to the corresponding state variables *S*, V, G, *E*, *A*, *I*, *H*, *R*, *B*. The proofs go as follows:
From (16 – 24)

$$\frac{d\lambda_1}{dt} = -\frac{\partial H_1}{\partial S} = \lambda_1\left((1-u_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(1-u_1)\left(\frac{\beta_B B}{K+B}\right)\right)+\lambda_1(\mu+mu_1+fu_2)-\lambda_2 mu_1$$
$$-\lambda_3 fu_2-\lambda_4\left((1-u_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(1-u_1)\left(\frac{\beta_B B}{K+B}\right)\right)$$

$$\frac{d\lambda_1}{dt} = (\lambda_1-\lambda_4)\left((1-u_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(1-u_1)\left(\frac{\beta_B B}{K+B}\right)\right)+(\lambda_1-\lambda_2)mu_1+(\lambda_1-\lambda_3)fu_2+\lambda_1\mu$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H_1}{\partial V} = -\lambda_1 n+(n+\mu)\lambda_2$$
$$\frac{d\lambda_2}{dt} = (\lambda_2-\lambda_1)n+\mu\lambda_2$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H_1}{\partial G} = -e\lambda_1+(e+\mu)\lambda_3$$
$$\frac{d\lambda_3}{dt} = (\lambda_3-\lambda_1)e+\mu\lambda_3$$

$$\frac{d\lambda_4}{dt} = -\frac{\partial H_1}{\partial E} = \lambda_4(wu_3+\mu)-\lambda_5(1-q)wu_3-\lambda_6 qwu_3$$
$$=(\lambda_4-\lambda_5)wu_3+(\lambda_5-\lambda_6)qwu_3+\lambda_4\mu$$

$$\frac{d\lambda_5}{dt}=-\frac{\partial H_1}{\partial A}=-P_1+\lambda_5(\eta+\psi+\mu)-\psi\lambda_7-\eta\lambda_8-\delta\lambda_9+\lambda_1(1-u_1)\beta_2 S-\lambda_4(1-u_1)\beta_2 S$$

$$\frac{d\lambda_5}{dt}=-P_1+(\lambda_5-\lambda_8)\eta+(\lambda_5-\lambda_7)\psi+(\lambda_1-\lambda_4)(1-u_1)\beta_2 S-\delta\lambda_9+\mu\lambda_5$$

$$\frac{d\lambda_6}{dt}=-\frac{\partial H_1}{\partial I}=-P_2+\lambda_6(\theta+\rho+d_1+\mu)-\theta\lambda_7-\rho\lambda_8+\lambda_1(1-u_1)\beta_1 S-\lambda_4(1-u_1)\beta_1 S-\varepsilon\lambda_9$$

$$\frac{d\lambda_6}{dt}=-P_2+(\lambda_6-\lambda_7)\theta+(\lambda_6-\lambda_8)\rho+(\lambda_1-\lambda_4)(1-u_1)\beta_1 S+(d_1+\mu)\lambda_6-\varepsilon\lambda_9$$

$$\frac{d\lambda_7}{dt}=-\frac{\partial H_1}{\partial H}=-P_3+\lambda_7(ru_4+d_2+\mu)-\lambda_8 ru_4+\lambda_1(1-u_1)\beta_3 S-\lambda_4(1-u_1)\beta_3 S-\gamma\lambda_9$$

$$\frac{d\lambda_7}{dt}=-P_3+(\lambda_7-\lambda_8)ru_4+(\lambda_1-\lambda_4)(1-u_1)\beta_3 S+(d_2+\mu)\lambda_7-\gamma\lambda_9$$

$$\frac{d\lambda_8}{dt}=-\frac{\partial H_1}{\partial R}=-\lambda_1\alpha_2(1-\phi)-\lambda_3\alpha_1\phi+\lambda_8(\alpha_2(1-\phi)+\alpha_1\phi+\mu)$$

$$\frac{d\lambda_8}{dt}=(\lambda_8-\lambda_1)\alpha_2(1-\phi)+(\lambda_8-\lambda_3)\alpha_1\phi+\mu\lambda_8$$

$$\frac{d\lambda_9}{dt}=-\frac{\partial H_1}{\partial B}=-P_4+\lambda_1(1-u_2)\left(\frac{K\beta_B S}{(K+B)^2}\right)-\lambda_4(1-u_2)\left(\frac{K\beta_B S}{(K+B)^2}\right)-\lambda_9(\sigma_3 u_5+\sigma_2-\sigma_1)$$

$$\frac{d\lambda_9}{dt}=-P_4+(\lambda_1-\lambda_4)(1-u_2)\left(\frac{K\beta_B S}{(K+B)^2}\right)-(\sigma_3 u_5+\sigma_2-\sigma_1)\lambda_9$$

Again, applying the necessary conditions from Pontryagin's maximum principle, we know that u^* must be a critical point of the Hamiltonian, that is, $\frac{\partial H_1}{\partial u_1}=\frac{\partial H_1}{\partial u_2}=\frac{\partial H_1}{\partial u_3}=\frac{\partial H_1}{\partial u_4}=\frac{\partial H_1}{\partial u_5}=0$. This leads to the following conditions on the optimal controls: $u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t)$ and $u_5^*(t)$, that is, the optimality equations are based on the conditions $\frac{\partial H_1}{\partial u_1}=\frac{\partial H_1}{\partial u_2}=\frac{\partial H_1}{\partial u_3}=\frac{\partial H_1}{\partial u_4}=\frac{\partial H_1}{\partial u_5}=0$.

Similarly, from (25 – 29)

$$\frac{\partial H_1}{\partial u_1}=Q_1u_1+\lambda_1(\beta_1 I+\beta_2 A+\beta_3 H)S-\lambda_1 mS+\lambda_2 mS-\lambda_4(\beta_1 I+\beta_2 A+\beta_3 H)S=0 \tag{30}$$

$$u_1^*=\frac{(\lambda_4-\lambda_1)(\beta_1 I+\beta_2 A+\beta_3 H)S+(\lambda_1-\lambda_2)mS}{Q_1}$$

$$\frac{\partial H_1}{\partial u_2}=Q_2u_2+\lambda_1\left(\frac{\beta_B B}{(K+B)}\right)S-\lambda_1 fS+\lambda_3 fS-\lambda_4\left(\frac{\beta_B B}{(K+B)}\right)S=0 \tag{31}$$

$$u_2^*=\frac{(\lambda_4-\lambda_1)\left(\frac{\beta_B B}{(K+B)}\right)S-(\lambda_1-\lambda_3)fS}{Q_2}$$

$$\frac{\partial H_1}{\partial u_3} = Q_3 u_3 - \lambda_4 wE + \lambda_5 (1-q) wE + \lambda_6 qwE = 0 \quad (32)$$

$$u_3^* = \frac{(\lambda_4 - \lambda_5) wE + (\lambda_5 - \lambda_6) qwE}{Q_3}$$

$$\frac{\partial H_1}{\partial u_4} = Q_4 u_4 - \lambda_7 rH + \lambda_8 rH = 0 \quad (33)$$

$$u_4^* = \frac{(\lambda_7 - \lambda_8) rH}{Q_4}$$

$$\frac{\partial H_1}{\partial u_5} = Q_5 u_5 - \lambda_9 \sigma_3 B = 0 \quad (34)$$

$$u_5^* = \frac{\lambda_9 \sigma_3 B}{Q_5}$$

Subject to the constraints: $0 \le u_1(t) \le u_{1\max}$, $0 \le u_2(t) \le u_{2\max}$, $0 \le u_3(t) \le u_{3\max}$, $0 \le u_4(t) \le u_{4\max}$, $0 \le u_5(t) \le u_{5\max}$ (Modnak, 2013).

Using the bounds of the control $u_1(t)$, its optimal control is given by

$$u_1^* = \min\left\{\max\left\{0, \frac{((\lambda_4 - \lambda_1)(\beta_1 I + \beta_2 A + \beta_3 H) + (\lambda_1 - \lambda_2) m) S}{Q_1}\right\}, 1\right\} \quad (35)$$

Using the bounds of the control $u_2(t)$, its optimal control is given by

$$u_2^* = \min\left\{\max\left\{0, \frac{\left((\lambda_4 - \lambda_1)\left(\frac{\beta_B B}{K + B}\right) + (\lambda_1 - \lambda_3) f\right) S}{Q_2}\right\}, 1\right\} \quad (36)$$

Using the bounds of the control $u_3(t)$, its optimal control is given by

$$u_3^* = \min\left\{\max\left\{0, \frac{(\lambda_4 - \lambda_5) wE + (\lambda_5 - \lambda_6) qwE}{Q_3}\right\}, 1\right\} \quad (37)$$

Using the bounds of the control $u_4(t)$, its optimal control is given by

$$u_4^* = \min\left\{\max\left\{0, \frac{(\lambda_7 - \lambda_8) rH}{Q_4}\right\}, 1\right\} \quad (38)$$

Using the bounds of the control $u_5(t)$, its optimal control is given by

$$u_5^* = \min\left\{\max\left\{0, \frac{\lambda_9 \sigma_3 B}{Q_5}\right\}, 1\right\}. \quad (35)$$

Therefore, from equations (35), (36), (37), (38) and (39), we obtain the following optimality system

$$\frac{dS}{dt}=\pi-\left\{1-\min\left\{\max\left\{0,\frac{((\lambda_4-\lambda_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(\lambda_1-\lambda_2)m)S}{Q_1}\right\},1\right\}\right\}(\beta_1 I+\beta_2 A+\beta_3 H)S$$
$$-\left\{1-\min\left\{\max\left\{0,\frac{\left((\lambda_4-\lambda_1)\left(\frac{\beta_B B}{K+B}\right)+(\lambda_1-\lambda_3)f\right)S}{Q_2}\right\},1\right\}\right\}\left(\frac{\beta_B B}{K+B}\right)S$$
$$-m\left\{1-\min\left\{\max\left\{0,\frac{((\lambda_4-\lambda_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(\lambda_1-\lambda_2)m)S}{Q_1}\right\},1\right\}\right\}S$$
$$-f\left\{1-\min\left\{\max\left\{0,\frac{\left((\lambda_4-\lambda_1)\left(\frac{\beta_B B}{K+B}\right)+(\lambda_1-\lambda_3)f\right)S}{Q_2}\right\},1\right\}\right\}S+nV+eG+\alpha_2(1-\phi)R-\mu S$$
$$\frac{dV}{dt}=-(n+\mu)V+\min\left\{\max\left\{0,\frac{((\lambda_4-\lambda_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(\lambda_1-\lambda_2)m)S}{Q_1}\right\},1\right\}mS$$
$$\frac{dG}{dt}=\alpha_1\phi R-(e+\mu)G+\min\left\{\max\left\{0,\frac{\left((\lambda_4-\lambda_1)\left(\frac{\beta_B B}{K+B}\right)+(\lambda_1-\lambda_3)f\right)S}{Q_2}\right\},1\right\}fS$$
$$\frac{dE}{dt}=\left\{1-\min\left\{\max\left\{0,\frac{((\lambda_4-\lambda_1)(\beta_1 I+\beta_2 A+\beta_3 H)+(\lambda_1-\lambda_2)m)S}{Q_1}\right\},1\right\}\right\}(\beta_1 I+\beta_2 A+\beta_3 H)S$$
$$+\left\{1-\min\left\{\max\left\{0,\frac{\left((\lambda_4-\lambda_1)\left(\frac{\beta_B B}{K+B}\right)+(\lambda_1-\lambda_3)f\right)S}{Q_2}\right\},1\right\}\right\}\left(\frac{\beta_B B}{K+B}\right)S$$
$$-\left\{\min\left\{\max\left\{0,\frac{(\lambda_4-\lambda_5)wE+(\lambda_5-\lambda_6)qwE}{Q_3}\right\},1\right\}\right\}wE-\mu E$$
$$\frac{dA}{dt}=(1-q)\left\{\min\left\{\max\left\{0,\frac{(\lambda_4-\lambda_5)wE+(\lambda_5-\lambda_6)qwE}{Q_3}\right\},1\right\}\right\}wE-(\eta+\psi+\mu)A$$
$$\frac{dI}{dt}=q\left\{\min\left\{\max\left\{0,\frac{(\lambda_4-\lambda_5)wE+(\lambda_5-\lambda_6)qwE}{Q_3}\right\},1\right\}\right\}wE-(\theta+\rho+d_1+\mu)I$$

$$\frac{dH}{dt} = \theta I + \psi A - \left\{ \min\left\{ \max\left\{ 0, \frac{(\lambda_7 - \lambda_8) rH}{Q_4} \right\}, 1 \right\} \right\} rH - (d_2 + \mu) H$$

$$\frac{dR}{dt} = \left\{ \min\left\{ \max\left\{ 0, \frac{(\lambda_7 - \lambda_8) rH}{Q_4} \right\}, 1 \right\} \right\} rH + \eta A + \rho I - \alpha_1 \phi R - \alpha_2 (1-\phi) R - \mu R$$

$$\frac{dB}{dt} = \varepsilon I + \delta A + \gamma H - \left\{ \min\left\{ \max\left\{ 0, \frac{\lambda_9 \sigma_3 B}{Q_5} \right\}, 1 \right\} \right\} \sigma_3 B - (\sigma_2 - \sigma_1) B$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H_1}{\partial S} = \lambda_1 \left((1-u_1)(\beta_1 I + \beta_2 A + \beta_3 H) + (1-u_1)\left(\frac{\beta_B B}{K+B} \right) \right) + \lambda_1 (\mu + mu_1 + fu_2) - \lambda_2 mu_1$$

$$- \lambda_3 fu_2 - \lambda_4 \left((1-u_1)(\beta_1 I + \beta_2 A + \beta_3 H) + (1-u_1)\left(\frac{\beta_B B}{K+B} \right) \right)$$

$$\frac{d\lambda_1}{dt} = (\lambda_1 - \lambda_4)\left(\left(1 - \left\{ \min\left\{ \max\left\{ 0, \frac{((\lambda_4 - \lambda_1)(\beta_1 I + \beta_2 A + \beta_3 H) + (\lambda_1 - \lambda_2) m) S}{Q_1} \right\}, 1 \right\} \right\} \right) (\beta_1 I + \beta_2 A + \beta_3 H) \right)$$

$$+ (\lambda_1 - \lambda_4)\left(\left(1 - \min\left\{ \max\left\{ 0, \frac{((\lambda_4 - \lambda_1)(\beta_1 I + \beta_2 A + \beta_3 H) + (\lambda_1 - \lambda_2) m) S}{Q_1} \right\}, 1 \right\} \right) \left(\frac{\beta_B B}{K+B} \right) \right)$$

$$+ (\lambda_1 - \lambda_2)\left\{ \min\left\{ \max\left\{ 0, \frac{((\lambda_4 - \lambda_1)(\beta_1 I + \beta_2 A + \beta_3 H) + (\lambda_1 - \lambda_2) m) S}{Q_1} \right\}, 1 \right\} \right\} m$$

$$+ (\lambda_1 - \lambda_3)\left\{ \min\left\{ \max\left\{ 0, \frac{\left((\lambda_4 - \lambda_1)\left(\frac{\beta_B B}{K+B} \right) + (\lambda_1 - \lambda_3) f \right) S}{Q_2} \right\}, 1 \right\} \right\} f + \lambda_1 \mu$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H_1}{\partial V} = -\lambda_1 n + (n + \mu)\lambda_2$$

$$\frac{d\lambda_2}{dt} = (\lambda_2 - \lambda_1) n + \mu \lambda_2$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H_1}{\partial G} = -e\lambda_1 + (e + \mu)\lambda_3$$

$$\frac{d\lambda_3}{dt} = (\lambda_3 - \lambda_1) e + \mu \lambda_3$$

$$\frac{d\lambda_4}{dt} = -\frac{\partial H_1}{\partial E} = \lambda_4 (wu_3 + \mu) - \lambda_5 (1-q) wu_3 - \lambda_6 qwu_3$$

$$= (\lambda_4 - \lambda_5) wu_3 + (\lambda_5 - \lambda_6) qwu_3 + \lambda_4 \mu$$

$$\frac{d\lambda_4}{dt} = (\lambda_4 - \lambda_5)\left\{ \min\left\{ \max\left\{ 0, \frac{(\lambda_4 - \lambda_5) wE + (\lambda_5 - \lambda_6) qwE}{Q_3} \right\}, 1 \right\} \right\} w$$

$$+\left(\lambda_5-\lambda_6\right)\left\{\min\left\{\max\left\{0,\frac{\left(\lambda_4-\lambda_5\right)wE+\left(\lambda_5-\lambda_6\right)qwE}{Q_3}\right\},1\right\}\right\}qw+\lambda_4\mu$$

$$\frac{d\lambda_5}{dt}=-\frac{\partial H_1}{\partial A}=-P_1+\lambda_5\left(\eta+\psi+\mu\right)-\psi\lambda_7-\eta\lambda_8-\delta\lambda_9+\lambda_1\left(1-u_1\right)\beta_2 S-\lambda_4\left(1-u_1\right)\beta_2 S$$

$$=-P_1+\left(\lambda_5-\lambda_8\right)\eta+\left(\lambda_5-\lambda_7\right)\psi+\left(\lambda_1-\lambda_4\right)\left(1-u_1\right)\beta_2 S-\delta\lambda_9+\mu\lambda_5$$

$$\frac{d\lambda_5}{dt}=-P_1+\left(\lambda_1-\lambda_4\right)\left(1-\left\{\min\left\{\max\left\{0,\frac{\left(\left(\lambda_4-\lambda_1\right)\left(\beta_1 I+\beta_2 A+\beta_3 H\right)+\left(\lambda_1-\lambda_2\right)m\right)S}{Q_1}\right\},1\right\}\right\}\right)\beta_2 S$$

$$+\left(\lambda_5-\lambda_8\right)\eta+\left(\lambda_5-\lambda_7\right)\psi-\delta\lambda_9+\mu\lambda_5$$

$$\frac{d\lambda_6}{dt}=-\frac{\partial H_1}{\partial I}=-P_2+\lambda_6\left(\theta+\rho+d_1+\mu\right)-\theta\lambda_7-\rho\lambda_8+\lambda_1\left(1-u_1\right)\beta_1 S-\lambda_4\left(1-u_1\right)\beta_1 S-\varepsilon\lambda_9$$

$$=-P_2+\left(\lambda_6-\lambda_7\right)\theta+\left(\lambda_6-\lambda_8\right)\rho+\left(\lambda_1-\lambda_4\right)\left(1-u_1\right)\beta_1 S+\left(d_1+\mu\right)\lambda_6-\varepsilon\lambda_9$$

$$\frac{d\lambda_6}{dt}=-P_2+\left(\lambda_1-\lambda_4\right)\left(1-\left\{\min\left\{\max\left\{0,\frac{\left(\left(\lambda_4-\lambda_1\right)\left(\beta_1 I+\beta_2 A+\beta_3 H\right)+\left(\lambda_1-\lambda_2\right)m\right)S}{Q_1}\right\},1\right\}\right\}\right)\beta_1 S$$

$$+\left(\lambda_6-\lambda_7\right)\theta+\left(\lambda_6-\lambda_8\right)\rho+\left(d_1+\mu\right)\lambda_6-\varepsilon\lambda_9$$

$$\frac{d\lambda_7}{dt}=-\frac{\partial H_1}{\partial H}=-P_3+\lambda_7\left(ru_4+d_2+\mu\right)-\lambda_8 ru_4+\lambda_1\left(1-u_1\right)\beta_3 S-\lambda_4\left(1-u_1\right)\beta_3 S-\gamma\lambda_9$$

$$=-P_3+\left(\lambda_7-\lambda_8\right)ru_4+\left(\lambda_1-\lambda_4\right)\left(1-u_1\right)\beta_3 S+\left(d_2+\mu\right)\lambda_7-\gamma\lambda_9$$

$$\frac{d\lambda_7}{dt}=-P_3+\left(\lambda_1-\lambda_4\right)\left(1-\left\{\min\left\{\max\left\{0,\frac{\left(\left(\lambda_4-\lambda_1\right)\left(\beta_1 I+\beta_2 A+\beta_3 H\right)+\left(\lambda_1-\lambda_2\right)m\right)S}{Q_1}\right\},1\right\}\right\}\right)\beta_3 S$$

$$+\left(\lambda_7-\lambda_8\right)\left\{\min\left\{\max\left\{0,\frac{\left(\lambda_7-\lambda_8\right)rH}{Q_4}\right\},1\right\}\right\}r+\left(d_2+\mu\right)\lambda_7-\gamma\lambda_9$$

$$\frac{d\lambda_8}{dt}=-\frac{\partial H_1}{\partial R}=-\lambda_1\alpha_2\left(1-\phi\right)-\lambda_3\alpha_1\phi+\lambda_8\left(\alpha_2\left(1-\phi\right)+\alpha_1\phi+\mu\right)$$

$$=-\lambda_1\alpha_2\left(1-\phi\right)-\lambda_3\alpha_1\phi+\lambda_8\alpha_2\left(1-\phi\right)+\lambda_8\alpha_1\phi+\lambda_8\mu$$

$$\frac{d\lambda_8}{dt}=\left(\lambda_8-\lambda_1\right)\alpha_2\left(1-\phi\right)+\left(\lambda_8-\lambda_3\right)\alpha_1\phi+\mu\lambda_8$$

$$\frac{d\lambda_9}{dt}=-\frac{\partial H_1}{\partial B}=-P_4+\lambda_1\left(1-u_2\right)\left(\frac{K\beta_B S}{\left(K+B\right)^2}\right)-\lambda_4\left(1-u_2\right)\left(\frac{K\beta_B S}{\left(K+B\right)^2}\right)-\lambda_9\left(\sigma_3 u_5+\sigma_2-\sigma_1\right)$$

$$=-P_4+\left(\lambda_1-\lambda_4\right)\left(1-u_2\right)\left(\frac{K\beta_B S}{\left(K+B\right)^2}\right)+\left(\sigma_3 u_5+\sigma_2-\sigma_1\right)\lambda_9$$

$$\frac{d\lambda_9}{dt}=-P_4+\left(\lambda_1-\lambda_4\right)\left(1-\left\{\min\left\{\max\left\{0,\frac{\left(\left(\lambda_4-\lambda_1\right)\left(\frac{\beta_B B}{K+B}\right)+\left(\lambda_1-\lambda_3\right)f\right)S}{Q_2}\right\},1\right\}\right\}\right)\left(\frac{K\beta_B S}{\left(K+B\right)^2}\right)$$

$$-\left(\sigma_3\left(\min\left\{\max\left\{0,\frac{\lambda_9\sigma_3 B}{Q_5}\right\},1\right\}\right)+\sigma_2-\sigma_1\right)\lambda_9$$

To determine the convexity of the system (that is to check whether the critical point is actually minimum), that is $\frac{\partial^2 H_1}{\partial u_i^2}>0$, we differentiate (30) – (34) w.r.t. $u_i; i=1,2,3,4,5$, we have $\frac{\partial^2 H_1}{\partial u_1^2}=Q_1>0$, $\frac{\partial^2 H_1}{\partial u_2^2}=Q_2>0$, $\frac{\partial^2 H_1}{\partial u_3^2}=Q_3>0$, $\frac{\partial^2 H_1}{\partial u_4^2}=Q_4>0$, $\frac{\partial^2 H_1}{\partial u_5^2}=Q_5>0$, where Q_1, Q_2, Q_3, Q_4 and Q_5 are nonnegative constant.

Numerical Simulation of the optimal control measure in the shigella model with results and interpretation

We conduct numerical simulation in order to investigate the effects of the control strategies on the transmission dynamics of Shigellosis. The simulations are performed using MALAB and we set time in days. The estimated initial values of the state variables of the model are $S_0=2000, V_0=120, G_0=150, E_0=40, A_0=90, I_0=30, H_0=60, R_0=200$ and $B_0=300$ and for the adjoint system we have terminal conditions $\{\lambda_i(T)\forall i=1,2,3,4,5,6,7,8,9\}=0$, where we set T = 50 days. The cost coefficients corresponding to state variables are estimated to be $P_1=0.01, P_2=0.05, P_3=0.02$ and $P_4=0.09$. The quadratic cost coefficient corresponding to control measures is estimated to be $Q_1=100, Q_2=50, Q_3=\frac{100}{3}, Q_4=25$ and $Q_5=20$. We perform numerical simulation of the optimality system by using the parameter values given in **Table 2** above.

We plot the graphs to show the effects of the control strategies: Vaccine, Education campaign, Screening, treatment and germicide in eradicating shigella infection. But we first plot the graphs when no control strategy is implemented and then compare their results with when the control strategies are implemented as shown in Figures 5(a)–5(d) in order to determine the best option to control the disease for maximum benefit.

The numerical solutions of the modified model with and without optimal control strategies

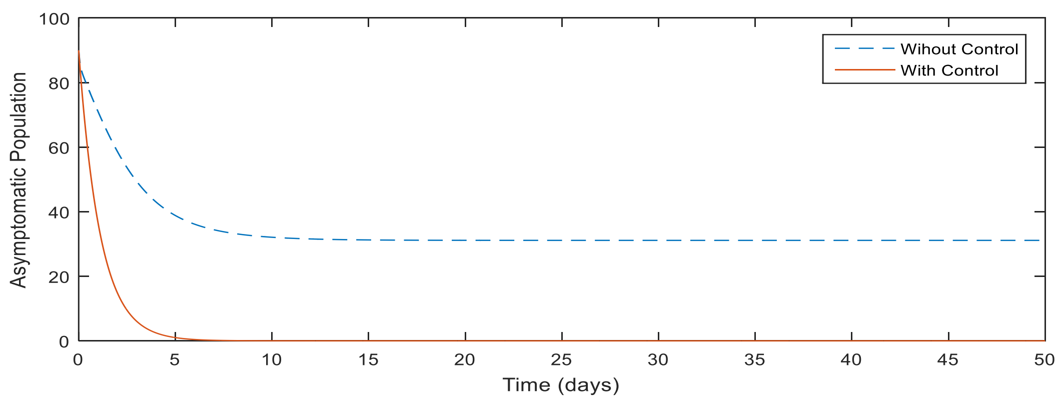

Figure 5(a) Comparison: Numerical solutions for the asymptomatic population with and without optimal control functions.

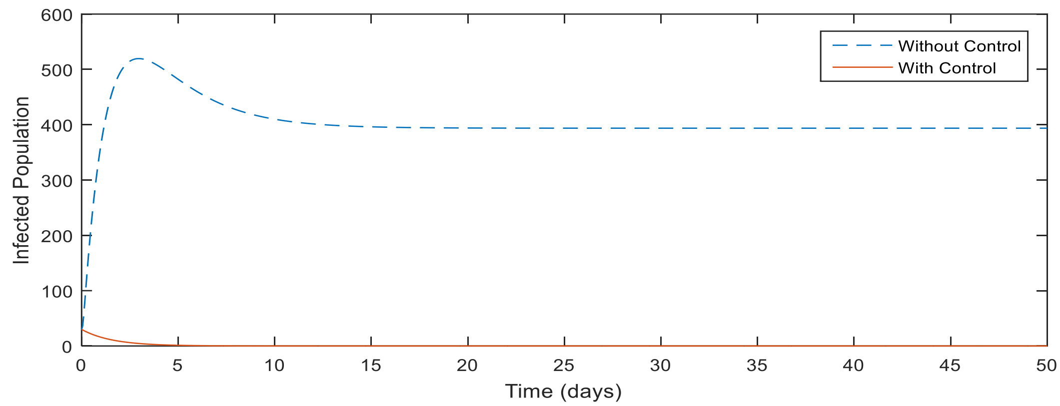

Figure 5(b) Comparison: Numerical solutions for the infected population with and without optimal control functions.

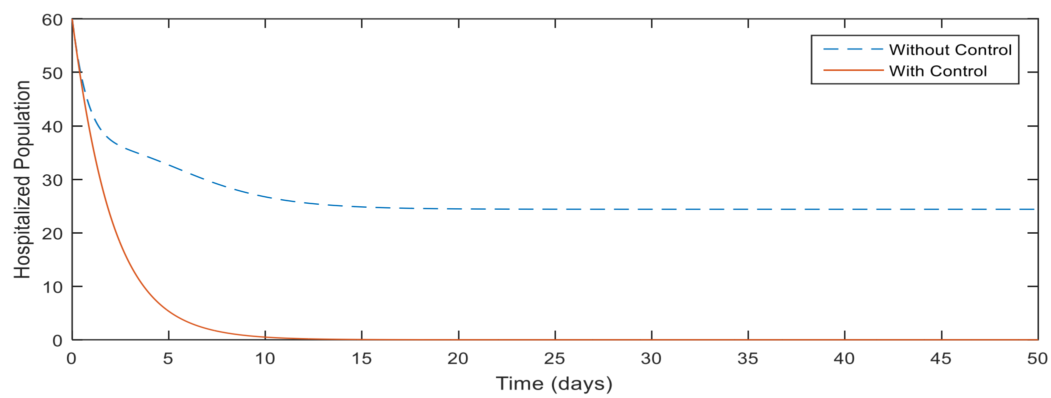

Figure 5(c) Comparison: Numerical solutions for the hospitalized population with and without optimal control functions.

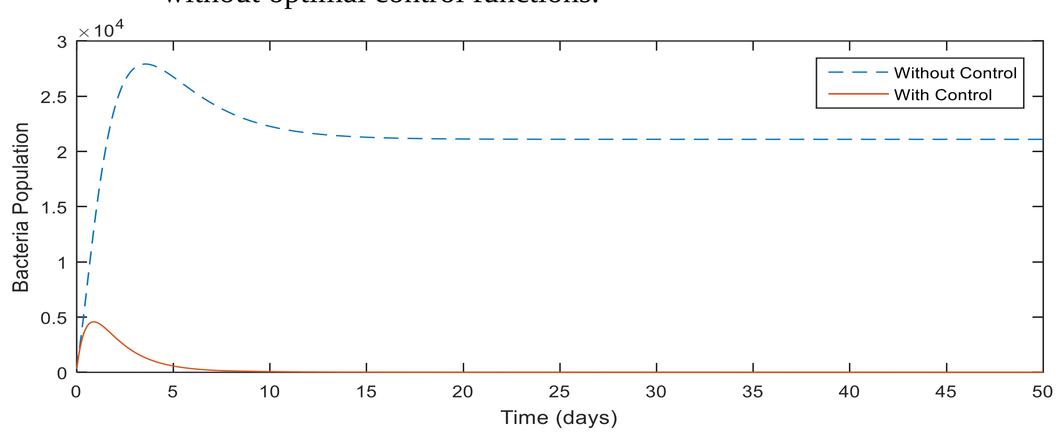

Figure 5(d) Comparison: Numerical solutions for the bacteria population with and without optimal control functions.

It can be seen from figures 5(a), 5(b), 5(c) and 5(d) that with the application of optimal control strategies: vaccination, education, screening, treatment and sanitation, there is a significant reduction in the number of exposed, asymptomatic, infectious, hospitalized and bacteria population at a given time. In the same vein, the application of the optimal control strategies helps to minimize shigella infection of the asymptomatic population after $t=5$ days, infected population after $t=3$ days, hospitalized population after $t=10$ days and bacteria population after $t=6$ days. With the application of multiple controls to contain shigella infection with respect to the result above, eradication of shigella infection is possible at the shortest period of time.

The Control profile of the model

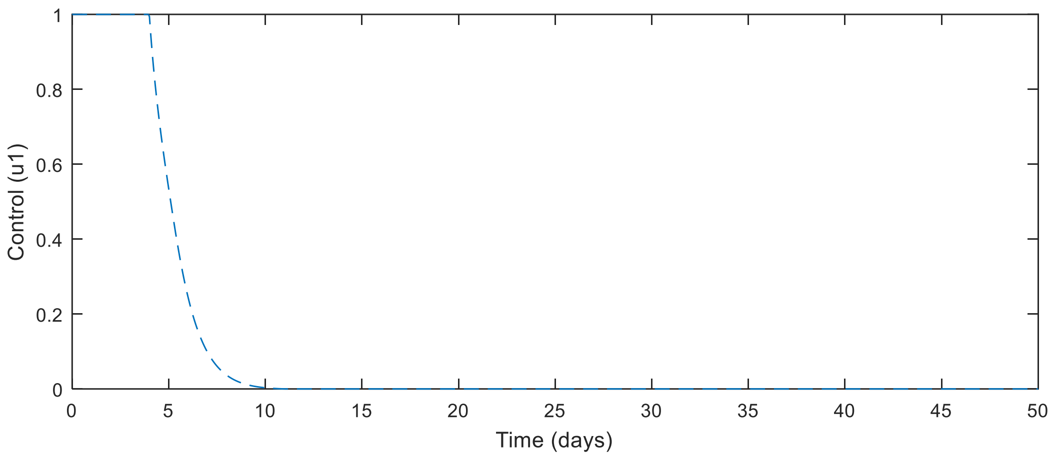

Figure 6: The profile of the control variable, vaccine (u_1)

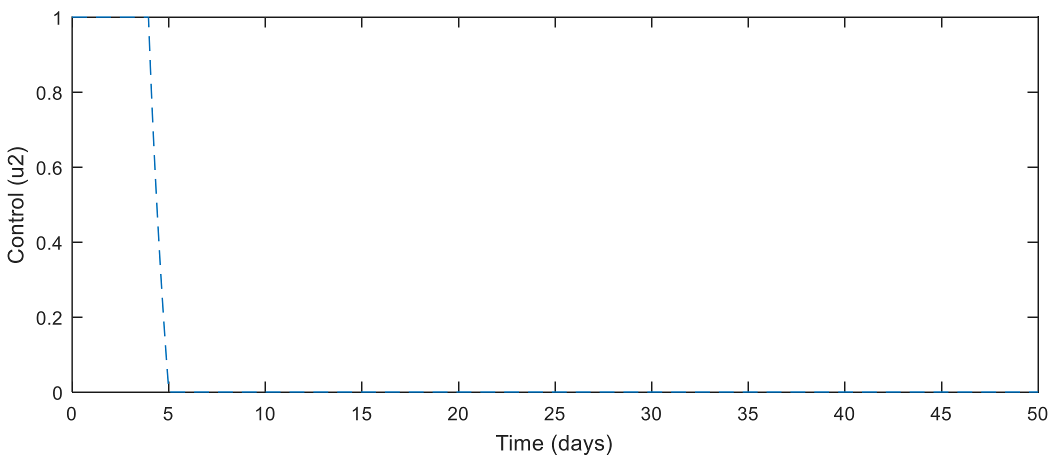

Figure 7: The profile of the control variable, education (u_2)

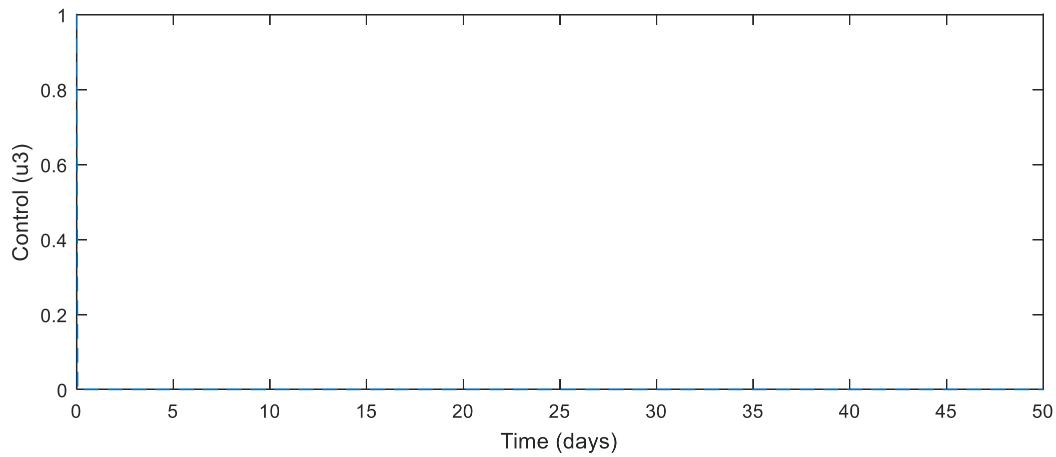

Figure 8: The profile of the control variable, screening (u_3)

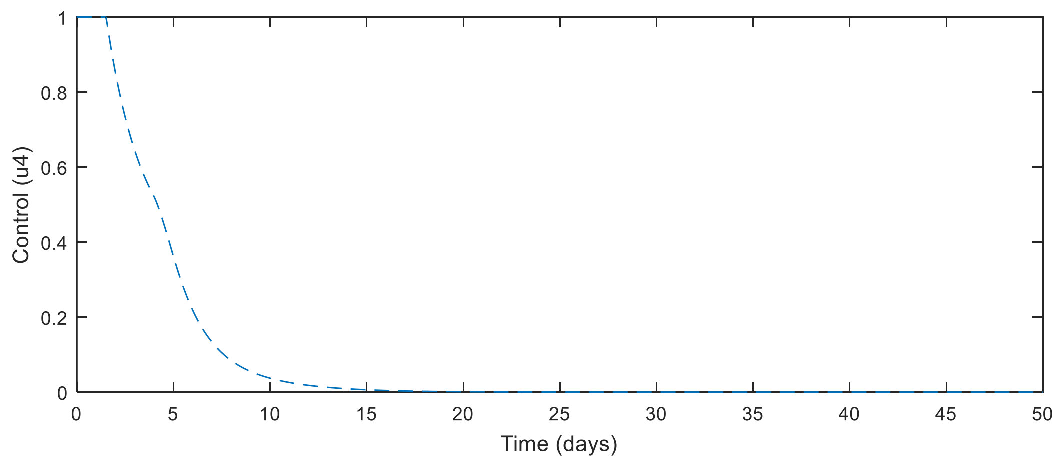

Figure 9: The profile of the control variable, treatment (u_4)

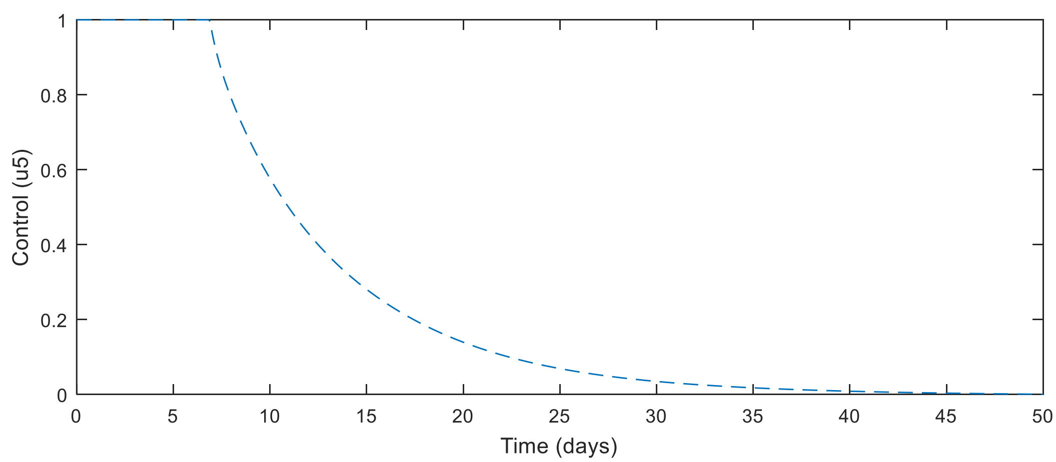

Figure 10: The profile of the control variable sanitation (u_5)

Interpretation of the Control Profile

It can be seen that Figure 6 to Figure10 are individual simulation of the time-dependent optimal controls (u1, u2, u3, u4, u5). The control u1 is at maximum level (i.e. $u1_{\max} = 1.0000$) at $t = 0$ and remains constant for $t = 4$ days before it gradually drops to its minimum level (i.e. $u4_{\min} = 0$) at the final time $t = 50$ days. The control u2 is at maximum level (i.e. $u1_{\max} = 1.0000$) at $t = 0$ and remains constant for $t = 4$ days before it sharply drops to its minimum level (i.e. $u4_{\min} = 0$) at the final time $t = 50$ days. The control u3 is at maximum level (i.e. $u3_{\max} = 1.0000$) at $t = 0$ and finally drops to its minimum level $u3_{\min} = 0$ at the final time $t = 50$ days. The control u4 is at the maximum level (i.e. $u4_{\max} = 1.0000$) at $t = 0$ and remains constant for $t = 1.5$ days before it gradually drops to its minimum level (i.e. $u4_{\min} = 0$) at the final time $t = 50$ days. The control u5 is at the maximum level (i.e. $u4_{\max} = 1.0000$) at $t = 0$ and remains constant for $t = 7$ days before it gradually drops to its minimum level (i.e. $u5_{\min} = 0$) at the final time $t = 50$ days.

The analysis and interpretation of the graph of the control values

When we use the controls u1, u2, u3, u4 and u5 to optimize the objective function, the following observations were made:

The control values are:

$u1 = 0.9999$

$u2 = 0.9999$

$u3 = 1.6841 \times 10^{-4}$

$u4 = 6.4646 \times 10^{-14}$

$u5 = 3.1588 \times 10^{-5}$

Proportion of $u1 = \dfrac{u1}{u1 + u2 + u3 + u4 + u5}$

$$u1 = \frac{0.9999}{2} \times 100 = 49.995\%$$

Proportion of $u2 = \dfrac{u2}{u1 + u2 + u3 + u4 + u5}$

$$u2 = \frac{0.9999}{2} \times 100 = 49.995\%$$

Proportion of $u3 = \dfrac{u3}{u1 + u2 + u3 + u4 + u5}$

$$u3 = \frac{1.6841}{2} \times 100 = 8.4206 \times 10^{-3}\%$$

Proportion of $u4 = \dfrac{u4}{u1 + u2 + u3 + u4 + u5}$

$$u4=\frac{6.4646\times10^{-14}}{2}\times100=3.2323\times10^{-12}\%$$

Proportion of $u5=\dfrac{u5}{u1+u2+u3+u4+u5}$

$$u5=\frac{3.1588\times10^{-5}}{2}\times100=1.5794\times10^{-3}\%$$

These results imply that for the shigella outbreak to be under control in the community, 49.995% of the vaccine, 49.995% of education campaign, $8.4206\times10^{-3}\%$ of screening, $3.2323\times10^{-12}\%$ of treatment and $1.5794\times10^{-3}\%$ of sanitation should be continually implemented.

CONCLUSION

Considering the result of study, we conclude that for optimal control of shigella disease, the emphasis should be on vaccine, public enlightenment and protection methods against the disease, followed by screening of infected individuals, environmental sanitation and then treatment in that order.

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